Efficient ℓ_∞ point triangulation through polyhedron collapse
Supplementary material

March 16, 2015

Part I

Proof of convergence: KKT criterion

1 Introduction

In the paper "Efficient ℓ_∞ point triangulation through polyhedron collapse" we have presented a novel algorithm for point triangulation based on the ℓ_∞ norm. Here we will show that the KKT criterion is fulfilled if and only if our algorithm terminates.

The proof necessarily consists of two elements:

- **Necessity:** If the KKT criterion is fulfilled, our algorithm terminates.
- **Sufficiency:** If our algorithm terminates, the KKT criterion is fulfilled.

We will split up each of the proofs into four cases based on the number of active inequalities: two, three, four, and more than four.

We start by repeating the KKT criterion for our specific scenario. We denote the gradients of the l^{th} constraint of camera k as $\boldsymbol{n}_{k,l}$. When N constraints are active, we index them with $i \in \{1, \ldots, N\}$.

$$\begin{aligned}
&\nexists \text{ improving direction} \\
&\iff \exists \boldsymbol{\lambda} \in \mathbb{R}^N \succeq \mathbf{0} : \\
&\qquad \mathbf{0} = \sum_i \lambda_i \boldsymbol{n}_i \\
&\qquad 1 = \sum_i \lambda_i
\end{aligned}$$

2 Two active inequalities

In this case, the termination criterion for the proposed algorithm is that the two corresponding gradients are eachother's inverse.

2.1 Necessity

If the KKT criterion is fulfilled, we have:

$$\begin{aligned} &\exists \lambda_1 \geq 0, \lambda_2 \geq 0 \\ &\mathbf{0} = \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 \\ &1 = \lambda_1 + \lambda_2 \end{aligned}$$

Rembember that these gradients are non-zero and unit vectors. Without loss of generality, we assume that λ_1 is larger than zero.

$$\begin{aligned} &\mathbf{0} = \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 \\ \implies &\lambda_1 \boldsymbol{n}_1 = -\lambda_2 \boldsymbol{n}_2 \\ \implies &\boldsymbol{n}_1 = \left(1 - \frac{1}{\lambda_1}\right) \boldsymbol{n}_2 \\ \implies &\boldsymbol{n}_1 = \lambda \boldsymbol{n}_2 \text{ where } \lambda < 0 \end{aligned}$$

As both vectors are unit vectors and λ is strictly negative, the vectors are eachother's inverse. □

2.2 Sufficiency

If $\boldsymbol{n}_1 = -\boldsymbol{n}_2$ then the choice $\lambda_1 = 1/2$ and $\lambda_2 = 1/2$ fulfills the KKT criterion. □

3 Three active inequalities

In this case, the termination condition in the proposed algorithm is that the three gradients are coplanar and that:

$$\begin{aligned} \boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 + \boldsymbol{n}_3) \leq 0 \\ \boldsymbol{n}_2 \cdot (\boldsymbol{n}_1 + \boldsymbol{n}_3) \leq 0 \\ \boldsymbol{n}_3 \cdot (\boldsymbol{n}_1 + \boldsymbol{n}_2) \leq 0 \end{aligned} \tag{1}$$

3.1 Necessity

From the KKT constraints we get

$$\begin{aligned} &\mathbf{0} = \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \lambda_3 \boldsymbol{n}_3 \\ \implies &\begin{cases} \lambda_1 \boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = -\lambda_2 - \lambda_3 \boldsymbol{n}_2 \cdot \boldsymbol{n}_3 \\ \lambda_1 \boldsymbol{n}_1 \cdot \boldsymbol{n}_3 = -\lambda_3 - \lambda_2 \boldsymbol{n}_2 \cdot \boldsymbol{n}_3 \end{cases} \\ \implies &\lambda_1 \boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 + \boldsymbol{n}_3) = -(\lambda_2 + \lambda_3)(1 + \boldsymbol{n}_2 \cdot \boldsymbol{n}_3) \end{aligned}$$

In case $\lambda_1 = 0$, it follows that $\boldsymbol{n}_2 = -\boldsymbol{n}_3$ and $\boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 + \boldsymbol{n}_3) = 0$. If this is not the case, we see that $\boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 + \boldsymbol{n}_3) \leq 0$. Mutatis mutandis for the two other inequalities from (1). □

3.2 Sufficiency

For brevity, we will use the notation $\boldsymbol{\mu}_{i,j} = \boldsymbol{n}_i \cdot \boldsymbol{n}_j$. Starting from the system

$$\begin{cases} \boldsymbol{\mu}_{1,2} + \boldsymbol{\mu}_{1,3} \leq 0 \\ \boldsymbol{\mu}_{1,2} + \boldsymbol{\mu}_{2,3} \leq 0 \\ \boldsymbol{\mu}_{1,3} + \boldsymbol{\mu}_{2,3} \leq 0 \end{cases} \tag{2}$$

We see that at least two of $\boldsymbol{\mu}_{1,2}$, $\boldsymbol{\mu}_{1,3}$ and $\boldsymbol{\mu}_{2,3}$ should be lower than or equal to zero. Without loss of generality, assume that $\boldsymbol{\mu}_{2,3} \leq \boldsymbol{\mu}_{1,3} \leq \boldsymbol{\mu}_{1,2}$ and thus $\boldsymbol{\mu}_{2,3} \leq 0$ and $\boldsymbol{\mu}_{1,3} \leq 0$.

Case 1: $\boldsymbol{\mu}_{2,3} = -1$

We can construct the required convex combination as $\frac{1}{2}\boldsymbol{n}_2 + \frac{1}{2}\boldsymbol{n}_3 = \mathbf{0}$.

Case 2: $\boldsymbol{\mu}_{2,3} \in \ ]-1, 0]$

We will now construct a convex combination of the three gradients that equals the null vector.

$$\begin{aligned} &\lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \boldsymbol{n}_3 = \mathbf{0} \\ \implies &\begin{cases} \lambda_1 + \lambda_2 \boldsymbol{\mu}_{1,2} = -\boldsymbol{\mu}_{1,3} \\ \lambda_1 \boldsymbol{\mu}_{1,2} + \lambda_2 = -\boldsymbol{\mu}_{2,3} \end{cases} \end{aligned} \tag{3}$$

The way in which this system was procured is valid if and only if the constructed convex combination of gradients is not orthogonal to both $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. However, for this to be the case (because the three gradients are coplanar) these two vectors have to be parallel. But this is not possible because $\boldsymbol{\mu}_{1,2} \in \ ]-1, 0]$.

Solving the system (3), we arrive at:

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \frac{1}{1 - \boldsymbol{\mu}_{1,2}^2} \begin{bmatrix} 1 & -\boldsymbol{\mu}_{1,2} \\ -\boldsymbol{\mu}_{1,2} & 1 \end{bmatrix} \begin{bmatrix} -\boldsymbol{\mu}_{1,3} \\ -\boldsymbol{\mu}_{2,3} \end{bmatrix}$$

Both coefficients are positive. We prove this for the first coefficient:

$$\lambda_1 = \frac{1}{1 - \boldsymbol{\mu}_{1,2}^2} \left(\boldsymbol{\mu}_{1,2} \boldsymbol{\mu}_{2,3} - \boldsymbol{\mu}_{1,3} \right)$$

From $\boldsymbol{\mu}_{1,3} + \boldsymbol{\mu}_{1,2} \leq 0$ and $\boldsymbol{\mu}_{2,3} \leq \boldsymbol{\mu}_{1,3}$ we know that

$$\begin{aligned} &0 \leq (\boldsymbol{\mu}_{2,3} - \boldsymbol{\mu}_{1,3})(\boldsymbol{\mu}_{1,3} + \boldsymbol{\mu}_{1,2}) \\ \implies &\boldsymbol{\mu}_{2,3} \boldsymbol{\mu}_{1,2} \geq \boldsymbol{\mu}_{1,3} + \boldsymbol{\mu}_{1,3}(\boldsymbol{\mu}_{1,3} - \boldsymbol{\mu}_{2,3} + \boldsymbol{\mu}_{1,2} - 1) \end{aligned}$$

Now, because $\boldsymbol{\mu}_{1,3} \leq 0$, $\boldsymbol{\mu}_{1,3} + \boldsymbol{\mu}_{1,2} \leq 0$ and $\boldsymbol{\mu}_{2,3} \geq -1$, we know that the second term on the right hand side is positive:

$$\boldsymbol{\mu}_{2,3} \boldsymbol{\mu}_{1,2} \geq \boldsymbol{\mu}_{1,3} \implies \lambda_1 \geq 0$$

Mutatis mutandis we can prove the sign of λ_1.

It is now straightforward to construct the convex combination:

$$\frac{1}{1 + \lambda_1 + \lambda_2} \left(\lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \boldsymbol{n}_3 \right) = \mathbf{0}$$

□

4 Four active inequalities

In case four inequalities are active, the proposed algorithm terminates if there is a triplet that fulfils that causes termination (per the rules for three active inequalities), or if

$$\begin{cases} \boldsymbol{n}_1 \cdot \boldsymbol{d}_{234} \leq 0 \\ \boldsymbol{n}_2 \cdot \boldsymbol{d}_{134} \leq 0 \\ \boldsymbol{n}_3 \cdot \boldsymbol{d}_{124} \leq 0 \\ \boldsymbol{n}_4 \cdot \boldsymbol{d}_{123} \leq 0 \end{cases} \tag{4}$$

In which $\boldsymbol{d}_{ijk}$ equals the improving direction constructed from the triplet $\{i, j, k\}$ as

$$\begin{aligned} \boldsymbol{d}'_{ijk} &= \boldsymbol{n}_1 \times \boldsymbol{n}_2 + \boldsymbol{n}_2 \times \boldsymbol{n}_3 + \boldsymbol{n}_1 \times \boldsymbol{n}_3 \\ s_{ijk} &= (\boldsymbol{n}_1 \cdot \boldsymbol{d}'_{ijk}) / \left\| \boldsymbol{n}_1 \cdot \boldsymbol{d}'_{ijk} \right\| \\ \boldsymbol{d}_{ijk} &= s_{ijk} \boldsymbol{d}'_{ijk} \end{aligned}$$

in case the triplet is not coplanar, or as the average of two of the triplet's vectors if these vectors are coplanar such that the scalar product of any of the triplet's vectors with its improving direction is strictly positive.

4.1 Necessity

In case one of the $\lambda_i = 0$ (without loss of generality, assume λ_4) the proof reduces to the proof given in Section 3.1.

Therefore, we can assume for the rest of this section that $\lambda_i > 0,\ \forall i$. We rewrite one of the elements of the KKT criterion as follows:

$$\begin{aligned} &\boldsymbol{0} = \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \lambda_3 \boldsymbol{n}_3 + \lambda_4 \boldsymbol{n}_4 \\ &\implies \lambda_1(\boldsymbol{n}_1 \cdot \boldsymbol{d}_{123}) + \lambda_2(\boldsymbol{n}_2 \cdot \boldsymbol{d}_{123}) + \lambda_3(\boldsymbol{n}_3 \cdot \boldsymbol{d}_{123}) + \lambda_4(\boldsymbol{n}_4 \cdot \boldsymbol{d}_{123}) \end{aligned} \tag{5}$$

As the construction of $\boldsymbol{d}_{123}$ differs depending on the coplanarity of the constiting vectors, we split up into the two possibilities.

The triplet $\{1, 2, 3\}$ is coplanar
In this case, $\boldsymbol{d}_{123}$ equals the average of two of the vectors in the triplet $\{1, 2, 3\}$. Without loss of generality, assume that $\boldsymbol{d}_{123} = \frac{1}{2}\boldsymbol{n}_1 + \frac{1}{2}\boldsymbol{n}_2$. From (5) we arrive at:

$$\implies \lambda_4\ \boldsymbol{n}_4 \cdot \boldsymbol{d}_{123} = -\frac{1}{2}(\lambda_1 + \lambda_2)(1 + \boldsymbol{n}_1 \cdot \boldsymbol{n}_2) - \frac{1}{2}\lambda_3(\boldsymbol{n}_1 \cdot \boldsymbol{n}_3 + \boldsymbol{n}_2 \cdot \boldsymbol{n}_3)$$

The two vectors from $\{1, 2, 3\}$ are chosen such that $\boldsymbol{n}_1 \cdot \boldsymbol{n}_3 + \boldsymbol{n}_2 \cdot \boldsymbol{n}_3 \geq 0$ and therefore the right hand side is non-positive. As $\lambda_4 > 0$ we conclude that $\boldsymbol{n}_4 \cdot \boldsymbol{d}_{123} \leq 0$.

The triplet $\{1, 2, 3\}$ is not coplanar
Continuing from (5) we arrive at:

$$\implies \boldsymbol{n}_4 \cdot \boldsymbol{d}_{123} = -(\lambda_1 + \lambda_2 + \lambda_3)\frac{\|\boldsymbol{n}_1 \cdot \boldsymbol{d}_{123}\|}{\lambda_4}$$

This step is possible because $\boldsymbol{n}_1 \cdot \boldsymbol{d}'_{123} = \boldsymbol{n}_2 \cdot \boldsymbol{d}'_{123} = \boldsymbol{n}_3 \cdot \boldsymbol{d}'_{123}$. This follows from the properties of the cross product and the triple product. As the right hand side is negative, so is $\boldsymbol{n}_4 \cdot \boldsymbol{d}_{123}$.

Mutatis mutandis we prove the other inequalities in (4). □

4.2 Sufficiency

If any of the triplets fulfil the termination requirements of a triplet, the algorithm terminates. Say, without loss of generality, that the triplet $\{1, 2, 3\}$ fulfils the requirements. In that case we can choose $\lambda_4 = 0$ and select λ_1 through λ_3 according to the procedure in Section 3.2. We will therefore assume throughout the rest of this section that none of the triplets fulfil the termination requirements on their own.

At least one triplet is coplanar

This is not possible: in this case the quadruplet fulfils its constraints only if a coplanar triplet fulfils its constraints. If a triplet is coplanar and does not fulfil its constraints, all of its vectors lie in the same half-plane and hence we can express one of them as a positively weighted combination of the other two.

Without loss of generality, say the triplet $\{1, 2, 3\}$ is coplanar and that we can express $\boldsymbol{n}_3 = \kappa_1 \boldsymbol{n}_1 + \kappa_2 \boldsymbol{n}_2$, $\kappa_i \geq 0$. This results in a conflict:

$$\boldsymbol{n}_3 \cdot \boldsymbol{d}_{124} = \kappa_1 \boldsymbol{n}_1 \cdot \boldsymbol{d}_{124} + \kappa_2 \boldsymbol{n}_2 \cdot \boldsymbol{d}_{124}$$

This expression should be negative because of (4), yet the right hand side is a strictly positively weighed sum of strictly positive values. Hence, it is not possible for a triplet to be coplanar and for the quadruplet to still fulfil its constraints without that triplet fulfilling its constraints as well.

None of the triplets is coplanar

We construct a convex combination as combination of the requirements

$$\boldsymbol{0} = \lambda_1 \boldsymbol{n}_1 + \lambda_2 \boldsymbol{n}_2 + \lambda_3 \boldsymbol{n}_3 + \lambda_4 \boldsymbol{n}_4$$

and

$$1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \tag{6}$$

From the first equality we derive four equalities by taking the scalar product of both sides with the improving directions of each of the possible triplets. This results in the following system:

$$\begin{bmatrix} \boldsymbol{n}_1 \cdot \boldsymbol{d}_{234} & \boldsymbol{n}_2 \cdot \boldsymbol{d}_{234} & \boldsymbol{n}_3 \cdot \boldsymbol{d}_{234} & \boldsymbol{n}_4 \cdot \boldsymbol{d}_{234} \\ \boldsymbol{n}_1 \cdot \boldsymbol{d}_{134} & \boldsymbol{n}_2 \cdot \boldsymbol{d}_{134} & \boldsymbol{n}_3 \cdot \boldsymbol{d}_{134} & \boldsymbol{n}_4 \cdot \boldsymbol{d}_{134} \\ \boldsymbol{n}_1 \cdot \boldsymbol{d}_{124} & \boldsymbol{n}_2 \cdot \boldsymbol{d}_{124} & \boldsymbol{n}_3 \cdot \boldsymbol{d}_{124} & \boldsymbol{n}_4 \cdot \boldsymbol{d}_{124} \\ \boldsymbol{n}_1 \cdot \boldsymbol{d}_{123} & \boldsymbol{n}_2 \cdot \boldsymbol{d}_{123} & \boldsymbol{n}_3 \cdot \boldsymbol{d}_{123} & \boldsymbol{n}_4 \cdot \boldsymbol{d}_{123} \end{bmatrix} . \boldsymbol{\lambda} = \boldsymbol{0}$$

Again we make a distinction depending on the coplanarity of the vectors: either at least one triplet is coplanar, or none of the triplets is coplanar.

Per the construction of the improving direction, the scalar product of a gradient and the improving direction of any of the triplets it is in is strictly positive. We divide each of the rows by the corresponding scalar product of the improving direction and one of the vectors in its triplet. The resulting system is:

$$\begin{bmatrix} \boldsymbol{n}_1 \cdot \boldsymbol{d}_{234} / \boldsymbol{n}_2 \cdot \boldsymbol{d}_{234} & 1 & 1 & 1 \\ 1 & \boldsymbol{n}_2 \cdot \boldsymbol{d}_{134} / \boldsymbol{n}_1 \cdot \boldsymbol{d}_{134} & 1 & 1 \\ 1 & 1 & \boldsymbol{n}_3 \cdot \boldsymbol{d}_{124} / \boldsymbol{n}_1 \cdot \boldsymbol{d}_{124} & 1 \\ 1 & 1 & 1 & \boldsymbol{n}_4 \cdot \boldsymbol{d}_{123} / \boldsymbol{n}_1 \cdot \boldsymbol{d}_{123} \end{bmatrix} . \boldsymbol{\lambda} = \boldsymbol{0}$$

Denote the entry on the diagonal for the i^{th} row as ϕ_i. We now subtract each of the rows from equation (6) in order to arrive at

$$\begin{bmatrix} 1-\phi_1 & 0 & 0 & 0 \\ 0 & 1-\phi_2 & 0 & 0 \\ 0 & 0 & 1-\phi_3 & 0 \\ 0 & 0 & 0 & 1-\phi_4 \end{bmatrix} \cdot \boldsymbol{\lambda} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

From (4) it follows that $\phi_i \leq 0,\ \forall i$. The required convex combination is acquired by normalisation of $\lambda_i = \frac{1}{1-\phi_i}$. □

4.3 More than four active inequalities

In case four inequalities are active, the proposed algorithm terminates if there is a triplet that fulfils that causes termination (per the rules for three active inequalities) or if for each triplet the corresponding improving direction has a negative scalar product with at least one of the gradients.

4.4 Necessity

If exactly three λ_i or non-zero, the corresponding triplet fulfils the constraints per section 3.1. We therefore assume more than three of them are non-zero.

Starting from the expression in the KKT criterion that

$$\sum_i \lambda_i \boldsymbol{n}_i = \boldsymbol{0}, \lambda_i \geq 0$$

For each triplet τ, we split up the summation:

$$\sum_{i \notin \tau} \lambda_i \boldsymbol{n}_i = -\sum_{i \in \tau} \lambda_i \boldsymbol{n}_i$$

The right hand side is non-positive: each gradient has a strictly positive scalar product with the improving direction of any triplet it belongs to and the weights are non-negative.

The left hand side, on the other hand, has at least one strictly positive weight (as more than three λ_i are non-zero). The left hand side must be non-positive, and all of its weights are non-negative (with at least one strictly positive weight). As a result, one of the terms being weighed must be non-positive. □

4.5 Sufficiency

We show that it is possible to reduce this case to the case of four active inequalities: it is always possible to choose four inequalities such that those four also fulfil their constraints.

In order to do so, we assume that there is an ordering on the set of triplets which we index with τ. We then construct a matrix A as in section (4.2) so that

$$(\mathrm{A}_{\tau,i}) = \boldsymbol{n}_i \cdot \boldsymbol{d}_\tau$$

The matrix R is constructed by concatenating a row of 1's at the bottom:

$$\mathrm{R} = \left[\frac{\mathrm{A}}{\mathbf{1}}\right]$$

The addition of the last row has increased the rank at most by 1. The rank of R is therefore at most 4.

Then the current case is equivalent to the case of four active inequalities, for which the proof is given in section (4.2).

Assume that the set $S_N = \{\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3, \boldsymbol{n}_4\}$ spans the column space of R (as each column corresponds to one of the gradients, and each row corresponds to one of the triplets).

Construct the matrix R_4 by selecting the four columns corresponding with S_N and the four rows corresponding with all possible triplets in this quadruplet in addition to the final row. Now we need only show that $\mathrm{rank}(\mathrm{R}_4) = \mathrm{rank}(\mathrm{R})$.

We can do this by showing that the set S_T of improving directions for the triplets formed from the gradients in S_N spans the set Ω_T of all improving directions from the possible triplets of Ω_N, the set of all gradients for the active inequalities.

Geometrically, the improving directions are the central lines (bisectors) of the cones defined by triplets formed from S_N. If the vectors in S_N are not coplanar (and hence, span $\mathbb{R}^3$), then the central lines are not coplanar either and hence also span $\mathbb{R}^3$. For the planar option, if the vectors are not colinear, the central lines are not colinear and span the plane. If all vectors are colinear, then the improving directions all lie on the same line and hence span this line. $\square$

Part II

The proposed cost function as MLE

We assume that

- the noise is additive white noise independent over the cameras characterized by the distribution n for each coordinate independently.
- the camera descriptions are accurately known.

Then the probability of making the set of observations $\{\tilde{\boldsymbol{u}}_1, \ldots, \tilde{\boldsymbol{u}}_K\}$ given the ground truth location $\boldsymbol{r}$ is

$$\begin{aligned} p\left(\tilde{\boldsymbol{u}}_1, \ldots, \tilde{\boldsymbol{u}}_K | \boldsymbol{r}\right) &= \prod_{k=1}^{K} p_k(\tilde{\boldsymbol{u}}_k | \boldsymbol{r}_k) \\ &= \prod_{k=1}^{K} n\left(\tilde{x}_k - \frac{X_k}{Z_k}\right) n\left(\tilde{y}_k - \frac{Y_k}{Z_k}\right) \end{aligned}$$

Assuming that the noise follows a uniform distribution with radius L we get:

$$n(x) \sim \begin{cases} \dfrac{1}{2L} & \text{if } |x| \leq L \\ 0 & \text{elsewhere} \end{cases}$$

We approach the radius of the distribution as an unknown that needs to be estimated as well. In that case, maximising the probability results in:

$$\max_{\boldsymbol{r},L} \prod_{k=1}^{K} n\left(\tilde{x}_k - \frac{X_k}{Z_k}\right) n\left(\tilde{y}_k - \frac{Y_k}{Z_k}\right)$$

As there are definitely values of L for which this probability is not zero, we wish to estimate the best such value and the corresponding point $\boldsymbol{r}$:

$$\begin{aligned} &\max_{\boldsymbol{r},L} \left(\frac{1}{2L}\right)^{2K} && \text{subject to } \left|\tilde{x}_k - \frac{X_k}{Z_k}\right| \leq L, \left|\tilde{y}_k - \frac{Y_k}{Z_k}\right| \leq L \\ \equiv &\min_{\boldsymbol{r},L} (L) && \text{subject to } \left|\tilde{x}_k - \frac{X_k}{Z_k}\right| \leq L, \left|\tilde{y}_k - \frac{Y_k}{Z_k}\right| \leq L \end{aligned}$$

This is exactly the cost function we arrive at by using the ℓ_∞ norm in both the reprojection error and the combined error. Note that the radius of the distribution is estimated by the value of γ in the paper.